UDC 539.375

STATIONARY MOTION OF A CRACK IN A STRIP

PMM Vol. 37, №3, 1973, pp. 572-576

V. D. KULIEV
(Moscow)

(Received April 7, 1972)

The stationary motion of a semi-infinite crack in a strip of finite width is considered. The motion of the crack in the strip is due to displacement of the rigidly clamped strip boundaries normally to the crack. The solution of the problem is constructed by the Wiener-Hopf method.

A similar problem has been examined in [1] in a static formulation. However, a computational error was committed in the factorization, whereupon the final formulas turn out to be in error.

Some stationary problems of the propagation of normal discontinuity cracks have been analyzed in [2 – 9]. In particular, it turns out that the Rayleigh velocity is an unachievable upper bound of the propagation velocity of a normal discontinuity crack.

1. The geometry to be investigated in this problem is shown in Fig. 1 with the corresponding coordinate system (the velocity of crack motion is constant). It is assumed that there exists a state of plane strain.

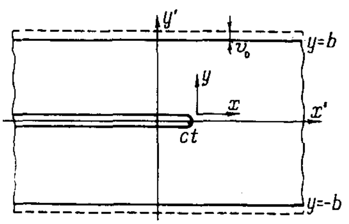

Fig. 1

The longitudinal and transverse wave potentials Φ and Ψ in the moving coordinate system $x = x' - ct$, $y = y'$ satisfy the following equations in stationary motion:

$$\beta^2 \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \quad \alpha^2 \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \quad (1.1)$$

$$(\beta = (1 - c^2 / c_1^2)^{1/2}, \quad \alpha = (1 - c^2 / c_2^2)^{1/2}, \ c < c_2)$$

Here c_1, c_2 are the longitudinal and transverse wave velocities in the elastic medium. By virtue of the symmetry relative to the x-axis it is sufficient to consider only the domain $0 \leqslant y \leqslant b$, $-\infty < x < \infty$. The boundary conditions in the moving coordinate system are

$$u = 0, \ v = v_0, \ -\infty < x < \infty \ \text{ for } \ y = b \quad (1.2)$$

$$\tau_{xy} = 0, \ \sigma_y = 0, \ x < 0 \ \text{ for } \ y = 0 \quad (1.3)$$

$$\tau_{xy} = 0, \ v = 0, \ x > 0 \ \text{ for } \ y = 0$$

$$(\Phi, \Psi) = O(r^{3/2}) \quad \text{for} \quad r = \sqrt{x^2 + y^2} \to 0 \quad \text{(condition on the edge [9])} \quad (1.4)$$

The solution of this problem can be represented as the sum of two stress fields. The first field corresponds to the homogeneous strain of a clamped infinite strip without a crack

$$\sigma_x = \frac{\nu E v_0}{(1+\nu)(1-2\nu)}, \quad \sigma_y = \frac{(1-\nu) E v_0}{(1+\nu)(1-2\nu)}, \quad \tau_{xy} = 0 \quad (1.5)$$

The second stress and displacement field is found from the solution of the following boundary value problem:

$$u = 0,\ v = 0,\ -\infty < x < \infty \quad \text{for } y = b$$

$$\left.\begin{aligned} \tau_{xy} &= 0,\ -\infty < x < \infty \\ v &= 0,\ x > 0 \end{aligned}\right\} \quad \text{for } y = 0 \tag{1.6}$$

$$\sigma_y = -\frac{(1-\nu) E v_0}{(1+\nu)(1-2\nu)}, \quad x < 0 \quad \text{for } y = 0$$

$$(\Phi, \Psi) = O(r^{3/2}) \quad \text{for } r = \sqrt{x^2 + y^2} \to 0 \tag{1.7}$$

2. Applying the exponential Fourier transform in the coordinate x to (1.1), the partial differential equations can be reduced to ordinary second order differential equations in y, whose solutions are

$$\Phi^* (s, y) = A(s) \exp(-s\beta y) + B(s) \exp(s\beta y) \tag{2.1}$$

$$\Psi^* (s, y) = C(s) \exp(-s\alpha y) + D(s) \exp(s\alpha y)$$

Here s is the complex transformation parameter, and the asterisk denotes the Fourier transform of the quantity indicated. Application of a Fourier transform in the coordinate x to the corresponding relationships for the stresses and displacements yields

$$\begin{gathered} \sigma_x^* = \lambda\Phi_{yy}^* - (\lambda + 2\mu) s^2\Phi^* - 2\mu is\Psi_y^* \\ \sigma_y^* = (\lambda + 2\mu)\ \Phi_{yy}^* - \lambda s^2\Phi^* + 2\mu is\Psi_y^* \\ \tau_{xy}^* = \mu(-2is\Phi_y^* + s^2\Psi^* + \Psi_{yy}^*) \\ u^* = -is\Phi^* + \Psi_y^*, \qquad v^* = \Phi_y^* + is\Psi^* \end{gathered} \tag{2.2}$$

According to boundary conditions (1.6) we have

$$\begin{gathered} (\sigma_y^*)_{y=0} = \sigma_+^* - \frac{(1-\nu) E v_0}{is (1+\nu)(1-2\nu)} \\ (v^*)_{y=0} = v_-^*, \qquad (\tau_{xy}^*)_{y=0} = 0 \\ (v^*)_{y=b} = 0, \qquad (u^*)_{y=b} = 0 \end{gathered} \tag{2.3}$$

Here

$$\sigma_+^* = \int_0^\infty (\sigma_y)_{y=0} e^{isx} dx, \qquad v_-^* = \int_{-\infty}^0 (v)_{y=0} e^{isx} dx$$

The constants of integration for the Fourier transforms of the longitudinal and transverse wave potentials (2.1) are determined from the boundary conditions (2.3). The last three of these conditions can be used to determine three constants in terms of the fourth. The determination of the fourth condition, while remaining constant, results in a Wiener-Hopf type equation to find the unknowns, the Fourier transforms of the normal stresses on the continuation of the crack σ_+^* and the vertical displacements of the crack lips v_-^*. Omitting the algebraic computations and lettig $F(s)$ denote the function

$$F(s) = \frac{s\{4\alpha\beta(1+\alpha^2) - \alpha\beta[(1+\alpha^2)^2 + 4]\operatorname{ch}(s\alpha b)\operatorname{ch}(s\beta b) + [(1+\alpha^2)^2 + 4\alpha^2\beta^2]\operatorname{sh}(s\alpha b)\operatorname{sh}(s\beta b)\}}{2(\alpha^2 - 1) M\beta\,[\alpha\beta\operatorname{sh}(s\beta b)\operatorname{ch}(s\alpha b) - \operatorname{sh}(s\alpha b)\operatorname{ch}(s\beta b)} \tag{2.4}$$

we arrive at the following equation of Wiener-Hopf type:

$$\frac{\sigma_+^*}{2\mu} - \frac{(1-\nu)\,Ev_0}{is2\mu\,(1+\nu)\,(1-2\nu)} = MF\,(s)\,v_-^* \tag{2.5}$$

$$\left(M = \frac{4\alpha\beta - (1+\alpha^2)^2}{2i\,(\alpha^2-1)\,\beta}\right)$$

Let us represent the function $F(s)$ as the product

$$F(s) = F_+(s)\,F_-(s) \tag{2.6}$$

Here the functions $F_+(s)$, $F_-(s)$ are analytic and different from zero in the upper and lower half-planes, respectively. Taking account of factorization, (2.5) can be written as:

$$\frac{\sigma_+^*}{2\mu F_+(s)} - \frac{(1-\nu)\,Ev_0}{is2\mu\,(1+\nu)\,(1-2\nu)\,F_+(s)} = MF_-(s)\,v_-^*$$

The second member in the left side of this equation has a first order pole at $s = 0$. Let us select the contour of integration in such a manner that the point $s = 0$ would belong to the upper half-plane of the s-plane. After an identical transformation, the equation of Wiener-Hopf type can be written in the form:

$$\frac{\sigma_+^*}{2\mu F_+(s)} - \frac{(1-\nu)\,Ev_0\,[F_+(0)-F_+(s)]}{is2\mu\,(1+\nu)\,(1-2\nu)\,F_+(0)\,F_+(s)} = MF_-(s)\,v_-^* + \frac{(1-\nu)\,Ev_0}{is2\mu\,(1+\nu)\,(1-2\nu)\,F_+(0)} \tag{2.7}$$

There remains to factorize the function $F(s)$ in the form (2.6). On the basis of the Weierstrass theorem on factorization, we can obtain

$$F_-(s) = F_-(0)\,\frac{1-s/s_0}{1-s/z_0}\prod_{n=1}^{\infty}\left\{\frac{(1-s/s_n)\,(1+s/\bar{s}_n)}{(1-s/z_n)\,(1+s/\bar{z}_n)}\right\} = F_+(-s)$$

$$\frac{z_0 e^{-\gamma/2}}{s_0\Gamma^2(^5/_4)}\prod_{n=1}^{\infty}\left\{\frac{|z_n|^2 e^{1/(2n)}}{|s_n|^2\,[1+1/(4n)]^2}\right\} = \frac{1}{F_-(0)}$$

$$F_-(0) = F_+(0) = e^{i\pi/4}\left[\frac{\beta\,(b)^{-1}\,(1-\beta^2)^{-1}\,(1-\alpha^2)^2}{4\alpha\beta-(1+\alpha^2)^2}\right]^{1/2}$$

Here γ is the Euler constant, $\Gamma(x)$ is the gamma function, and the complex numbers s_n and z_n are roots of the equations

$$4\alpha\beta\,(1+\alpha^2) - \alpha\beta\,[(1+\alpha^2)^2+4]\,\mathrm{ch}\,(s_n\alpha b)\,\mathrm{ch}\,(s_n\beta b) + [(1+\alpha^2)^2 + 4\,\alpha^2\beta^2]\times \mathrm{sh}\,(s_n\alpha b)\,\mathrm{sh}\,(s_n\beta b) = 0$$

$$\alpha\beta\,\mathrm{sh}\,(z_n\beta b)\,\mathrm{ch}\,(z_n\alpha b) - \mathrm{sh}\,(z_n\alpha b)\,\mathrm{ch}\,(z_n\beta b) = 0$$

Let us now apply the standard Wiener-Hopf procedure to (2.7). The left side of this equation is an analytic function in the upper half-plane of the s-plane, and the right side is an analytic function in the lower half-plane of the s-plane. By the principle of continuous extension it can be asserted that the left and right sides of this equation are the analytic continuations of each other. There remains to clarify the behavior of a function defined in such a manner, and analytic in the whole s-plane, at infinity. Using a theorem of Abelain type [10] and the condition at the "edge" (1.4), it can be shown that the analytic function tends to zero at infinity. Let us note that $F_+ s = s^{1/2}$ as $s \to \infty$. Then by virtue of the Liouville theorem, it is zero identically in the whole s-plane.

Therefore, we obtain

$$\sigma_+{}^* = \frac{i(1-\nu)Ev_0}{(1+\nu)(1-2\nu)s}\left[\frac{F_+(s)}{F_+(0)} - 1\right] \tag{2.8}$$

$$v_-{}^* = \frac{i(1-\nu)Ev_0}{2\mu(1+\nu)(1-2\nu)MF_+(0)}\frac{1}{sF_-(s)} \tag{2.9}$$

Now, let us determine the coefficient of stress intensity K which is of fundamental interest in the mechanics of brittle fracture. According to (2.15) we find as $s \to \infty$

$$\sigma_+{}^* = \frac{i(1-\nu)Ev_0}{(1+\nu)(1-2\nu)F_+(0)} s^{-1/2} \tag{2.10}$$

On the other hand, using the condition

$$\sigma_y = K / \sqrt{2\pi x} \quad \text{for} \quad x \to +0$$

at the end of the crack, we have

$$\sigma_+{}^* = \int_0^\infty (\sigma_y)_{y=0} e^{isx}, \qquad dx = \frac{K}{\sqrt{2}} e^{i\pi/4} \; s^{-1/2} \tag{2.11}$$

It is hence assumed that s tends to infinity while remaining in the upper half-plane. According to (2.10) and (2.11), we finally find the coefficient of stress intensity

$$K = \frac{Ev_0(b)^{1/2}}{(1+\nu)m}\left[\frac{(1-\nu)}{(1-2\nu)}\left(1 - \frac{1-2\nu}{2-2\nu}m^2\right)^{-1/2} R(m,\nu)\right]^{1/2} \tag{2.12}$$

Here

$$R(m,\nu) = 4\left[(1-m^2)\left(1 - \frac{1-2\nu}{2-2\nu}m^2\right)\right]^{1/2} - (2-m^2)^2$$

$$m = c / c_2$$

In particular, as $m \to 0$ we obtain the solution of the static problem

$$K^\circ = \frac{Ev_0\, b^{1/2}}{(1+\nu)(1-2\nu)^{1/2}}$$

It follows from (2.12) that as the crack propagation velocity increases, the coefficient of stress intensity drops monotonely and vanishes at the Rayleigh velocity $m = m_R$. For $m > m_R$ the coefficient of stress intensity becomes imaginary.

Formula (2.12) has been found in [9] by another method.

BIBLIOGRAPHY

1. Knauss, W. G., Stresses in an infinite strip containing a semi-infinite crack. Trans. ASME, Ser. E, J. Appl. Mech., Vol. 33, №2, 1966.
2. Yoffe, E. H., The moving Griffith crack. Philos. Mag. Vol. 42, №330, 1951.
3. Craggs, I. W., On the propagation of a crack in an elastic-brittle material. J. Mech. and Phys. Solids, Vol. 8, №1, 1960.
4. Barenblatt, G. I. and Cherepanov, G. P., On the wedging of brittle bodies. PMM Vol. 24, №4, 1960.
5. Cotterel, B., On the nature of moving cracks. Trans. ASME, Ser. E, J. Appl. Mech., Vol. 31, №1, 1964.
6. Gol'dshtein, R. V., Rayleigh waves and resonance phenomena in elastic bodies. PMM Vol. 29, №3, 1965.
7. Gol'dshtein, R. V., Stationary crack motion in a strip. Ultimate crack velocity. Inzh. Zh., Mech. Tverd. Tela, №2, 1968.
8. Gol'dshtein, R. V. and Matchinskii, M., On stationary crack motion in a strip. Inzh. Zh., №4, 1967.
9. Cherepanov, G. P., Mechanics of Brittle Fracture. "Nauka", Moscow, 1973.
10. Noble, B., Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations. Pergamon Press, 1959.

Translated by M. D. F.